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WHAT COULD THE LEAST INCONSISTENT NUMBER BE?

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1. *Introduction*

In 'Is Arithmetic Consistent', (Priest [1994]; hereafter IAC) I argued that the view that the set of truths of arithmetic is inconsistent has more going for it than might have been thought. Specifically, I compared the theory, N, of truths in the "standard model" of arithmetic with an inconsistent theory, N_n, and argued that N_n has a lot to be said for it. The n here refers to a number; specifically, the least inconsistent number, that is, the least number such that $n = n+1$. (If there is an inconsistent number then there is a least; moreover, if $n = n+1$ then for any $n+m = n+m+1$, and hence any larger number is inconsistent.)

A natural question that will have occurred to anyone who read that paper is 'what is n?' Beyond some remarks to the effect that n must be extraordinarily large, I made no attempt to address this question. And I might say right away that I can give no answer to it in the form 'n is so and so' where 'so and so' is a phrase employing only numerals and standard arithmetic operations. Nor do I think that an illuminating answer is likely to be forthcoming.

One might try to turn this fact into an objection against the plausibility of inconsistent arithmetic. I do not, myself, see it in that way. The fact that it is impossible to answer a question is no kind of objection unless there is reason to suppose that, on the theory in question, there ought to be some grounds for knowing the answer. As far as I can see, there are none, as far as the account in *IAC* goes.

Despite this, the question is an intriguing one. Moreover, the question has a certain embarrassment value for the line run in *IAC*. Though one might not legitimately be required to answer the question, it ought at least to be possible to say what it is that determines that n has this exceptional property. And, moreover, do this with the intellectual equivalent of a straight face. This is where the embarrassment lies.

In this paper, I want to explore various possibilities. (I make no claim that these are the only possibilities, though they seem to me to be the most interesting ones.) I will arrive at what I find to be plausible answers. However, the discussion is hardly definitive.

2. *Platonism*

Any answer to the question of what determines n to be the least inconsistent number is likely to be grounded in an account of the nature of numbers. As far as I can see, there are three accounts of the nature of number that provide some hope of giving an answer: platonism, empiricism and nominalism. I will discuss each of these in turn.

The platonist answer to the question at issue is very blunt. The language of arithmetic describes a certain abstract (= immaterial) mathematical structure; and as a matter of brute (platonic) fact, n just *is* the least inconsistent number. Since the structure is immaterial, it is natural to suppose that there are problems of epistemic access. This answer at least explains why, therefore, we have difficulty in answering the question of what n is. Moreover, if someone is prepared to accept platonism in the first place, I know of no way of showing this answer to be wrong.

Despite this, I find the answer an implausible one. For a start, I suspect that platonists are unlikely to find the claim that the inconsistent interpretation of arithmetic is the correct one as appealing as non-platonists. (Though this is a sociological comment. I see no specifically platonist *reasons* that they could offer.) More importantly in the present context, the flavour of *IAC* is strongly anti-platonist. It does not argue against platonism as such, though if the considerations it advances are correct, the idea of an interpretation of the language of mathematics as an abstract structure (in the platonist sense) is quite otiose; and its mystification falls to a swift stroke of Ockham's razor. So let us move on.

3. *Empiricism*

The next view of the nature of numbers which offers some hope of answering the question is one that grounds the nature of numbers in the empirical universe. Suppose that numbers are grounded in empirical objects, in some sense, and that there is a finite number of these, then there may be a finite number of numbers, and so a greatest. If there are reasons to suppose that this number, or maybe the next, the least non-existent number, behaves inconsistently, we have what we are after.

The view that numbers supervene in some sense on physical objects is often associated with the name of Mill, and is usually reckoned to have been dealt with decisively by Frege in the *Grundlagen* It is not so often observed that the account, in effect, reappears in Russell and Whitehead's *Principia*. In this, numbers are defined as collections of sets in one to one correspondence with each other. The sets in question (at least at the lowest level) are sets of objects of type zero. And objects of type zero are usually thought of as physical objects. Now if there is a finite number of these, there is a maximum number, or at least a maximum non-empty number. Russell, of course, realised this, and had to add the axiom of infinity to the machinery of *Principia* to rule out this possibility, though he never seemed very happy with it. (See, e.g., p 179 of Russell [1908] in van Heijenoort [1967].)

Maybe, then, Russell was wrong. There is a maximum (non-empty) number determined by the size of the physical cosmos. This at least gets us a non-arbitrarily selected number. Why should it mark the limit of consistency, however? On Russell's account, of course, it does not. Both the largest non-empty number and the smallest empty number (i.e., the empty set) behave quite consistently. We could, however, give the account a small twist that would help. Let us define numbers by descriptions. If m is a Russell-number, let us define the true cardinal of size m as $\iota x(x \neq \varnothing \wedge x = m)$. This has no effect for non-empty numbers; but, now, empty numbers turn into cases of reference failure. If there are only 36 objects in the universe, it seems at least as plausible (given that we are grounding numbers in it) to suppose that '37' is "undefined" as to suppose that it refers to the empty set.

The question now becomes why improper descriptions should behave inconsistently. A number of writers (e.g. Burge [1991]) have argued that if t is a non-denoting phrase then $t \neq t$ is true. Since this is precisely one of the properties of n, we might appear to be on the right lines. Unfortunately, we are not. The rationale for this behaviour is a claim that (at least extensional) atomic predicates appended to non-denoting terms produce falsehoods. If this is right then there is no way of getting $n = n+1$ to be true, as is also required.

Another approach to non-denotation, advocated by Scott [1991], is to require every term that does not denote naturally to denote a particular non-existent entity (i.e., an entity outwith the domain of quantification). This approach looks more promising. Given a mathematical structure with an undesirable gap, it is a common enough practice to invent an "ideal entity" to fill the gap. Thus, the point at infinity is often invoked to complete Euclidean geometry in a certain sense. Specifically, it ensures that the claim that all pairs of parallel lines meet at exactly one point is true. Now, a finite sequence of numbers is incomplete in an obvious sense. Hence we might well take it that there is an ideal (non-existent) entity, $*$, that provides the completion, specifically, by making the claim that every number has a unique successor true. Thus, all numerals greater than or equal to that for n would denote $*$, and hence we would have that $n = n+1$, etc.

This will give us part of what we want. But why, however, should $*$ behave inconsistently? Why should $n \neq n+1$ also? There is no immediate reason why it should. (Indeed, on Scott's semantics $*$ behaves quite consistently.) However, in the option we are contemplating, some things *do* behave inconsistently. And if so, $*$ would seem a pretty natural candidate for such behaviour. For a start, non-existent objects may (notoriously) be inconsistent (the round square, Dr Watson's war wound). Moreover, in the present case there is some reason to suppose that $*$ is inconsistent. One of the properties that one would expect any completion of finite arithmetic to have is precisely that every number is distinct from its successor. Hence we must have $*$ being distinct from its successor, which is, of course, $*$. Thus, $n = n+1$ is false.

4. *The Size of the Cosmos*

The above reasoning is hardly mandatory, of course. However, we are not in the process of arguing that arithmetic *is* inconsistent. That is being assumed in the present context. The point is to understand how this could be so. And the above reasoning would seem, at least, to suffice for that.

Despite this, I do not think that the account is satisfactory. The major reason is that this account of number cannot, in the end, support the conclusion that there is a maximum number. Although there is a maximum (non-empty) number in the *Principia* account of number being considered here, this fact is relative to type. The numbers of the next higher type will have a different, in fact, larger, maximum. Thus there is, in a sense, no maximum number. More precisely, for any cardinality, there are sets of greater cardinality, provided we can ascend the type hierarchy as far as we wish. It is only the restrictions of type theory that prevent us collecting these together to provide an infinite set, as is done in, say, *ZF* set theory (where the objects of type zero can be dispensed with altogether).

Even setting these problems aside, the existence of a maximum number is dependent on the claim that the class of physical (type zero) objects is finite; and I just do not think that this is true. There might be a finite number of fundamental particles (past present and future) in the cosmos, but there is an infinitude of points of space-time. Maybe one might insist that space and time are discrete, or that space time points are not real existents, or something of the kind. But it still remains true that part of an object is an object. Hence if there were even one physical object extended in space there would be an infinite number of objects: its parts. It might be replied that if the objects are fundamental particles they are not divisible. They may not be divisible physically, but there is no problem about dividing them conceptually. The left hand side of an object, even though it could not exist physically on its own, is still a perfectly good object conceptually.

Hence, it would seem, an account of the kind we have been looking at cannot provide what is required. So let us move on to another possibility: nominalism.

5. *Nominalism*

Nominalism dispenses with talk of numbers altogether, in favour of talk of numerals. Unlike numbers, it is clear that the collection of numerals available in practice is finite, and so that there is a maximum. Clearly, we are in the right ball-park. Traditionally, the finitude of numerals has been at the root of standard arguments against nominalism. There just don't seem to be enough to support all of arithmetic. Once numerals are allowed to behave inconsistently, however, this objection lapses, as the inconsistent arithmetics, in effect, demonstrate; and hence we may explore this possibility with an easier conscience.

The following account draws heavily on van Bendegem [1987] and also on Priest [1983]. In the simplest possible numeral system each numeral is a collection of strokes. (One stroke for 0, two for 1, etc.) Assuming that each stroke must have a minimal size, which is reasonable in practice — however broadly that notion is defined— then there can be only a finite number of physically possible numerals. The arithmetic operations of successor, addition, multiplication, etc, can be defined on numerals in the obvious way. E.g. the successor of two strokes is three strokes, etc. A problem arises as to how to handle arithmetic operations that take us, intuitively, beyond the greatest numeral. We will come back to this in a moment. The identity relation between arithmetic terms is defined in some suitable way; and once truth has been defined for identity statements, the truth of compound statements can be specified using truth functions and substitutional quantification.

Of course, there are other notation systems for arithmetic (such as the decimal system). And other systems may have numerals that are bigger than any numeral in the simple system. But whatever the system, there will, for the same reason as before, be a largest numeral.

Now let us turn to the question of operations that take us, intuitively, beyond the greatest numeral. Consider, in particular, the successor operation, s (adding a stroke). It is clear that in some sense, applying this operation to a numeral ought to produce another; but it is also true that this operation breaks down at the largest numeral. Suppose that we take $s(m) = *$ to represent 'the operation s applied to the numeral m is undefined). Then $s(m) = *$ iff m is the largest numeral, l.

The fact that the operation breaks down at l indicates that the system is incomplete in an obvious sense. And this suggests an act of completion.

We cannot add an infinitude of new symbols, but there is nothing to stop us taking '$*$' as a symbol in its own right. Rather like the point at infinity, this symbol has to represent the completion of the partial structure.

What arithmetic properties should it have? For a start $s(*)=*$. (Undefined input give undefined output. Next, for any (regular) numeral, m, $m \neq *$. Moreover, it is not implausible to suppose that $* \neq *$ (as well, of course, as that $*=*$). In a sense, $*$ must represent all inexpressible numerals, that is, it must have the properties that any of them has; but one property that each has is being different from another. Hence $*$ must be different from itself. This is precisely the idea behind the construction of the models that give rise the inconsitent arithmetics. Unsurprisingly, then, the truths of arithmetic generated by this account are just those in N_{I+1}.

As with the previous account, the story we have just been through is hardly mandatory. However, let me say again, we are not in the process of arguing that arithmetic *is* inconsistent. The point is to understand how it could be so. And the above reasoning would seem to suffice for that. In particular, it tells us what the greatest consistent numeral is: the greatest numeral that can be generated given a particular notational system for arithmetic.

6. N_ω

We have found an account that gives some explanation of why there should be a least inconsistent number (or, at least, numeral). The story is not over yet, however. As is clear, the greatest consistent number is relative to a notation system for arithmetic. An obvious question at this point is: when we talk about the truths of arithmetic, which notation system are we talking about? The easy reply to this is, of course: the one we actually employ (at the moment). However, this reply may be felt to be somewhat unsatisfactory. The notation system we use is a contingent thing. It has changed over history, and will doubtless change further. Does this mean that arithmetic truth changes? According to the account in question, the answer is 'yes'. Nor is it clear that there is anything too objectionable about this. As long as the greatest numeral possible with our notation system is so large that it has no physical or psychological significance —as surely the greatest numeral in standard notation is— changes will be of no moment at all.

But still, there may be some feeling of dissatisfaction. The truths of arithmetic are, after all, supposedly, necessary truths; and they should not depend on the vicissitudes of what notation we use, how small the symbols we can perceive are, or how much space there is to write them in. All these are surely too contingent. If this reasoning persuades you, there is another option at this point. We can take it that the truths of arithmetic are what is common to *all* notational systems. In particular, it is not difficult to show that if $i \leq k$ then $N_i \supseteq N_k$, i.e., the bigger i is the smaller is the corresponding set of truths. And thus the truths of arithmetic are just the sentences in the limit of this chain, $N_\omega = \cap\{N_n; n < \omega\}$. (Clearly, there is no maximal notational system. For any system we can introduce new symbols that represent combinations of symbols in the old system.)

N_ω is, in fact, an interesting theory in its own right. It has the following properties. The proof I defer to the technical appendix, section 8.

- i) $N_\omega \supseteq N$, and so N_ω is complete (i.e., for every sentence φ, N_ω contains either φ or $\neg\varphi$).
- ii) N_ω is a theory in the paraconsistent logic *LP*.
- iii) N_ω is inconsistent.
- iv) If φ is any (negated) equation then $\varphi \in N_\omega$ iff $\varphi \in N$. (Hence N_ω is "unsaturated", i.e., there is a formula of the form $\exists x\varphi(x)$ in N_ω and no formula of the form $\varphi(n)$.)
- v) N_ω is π_1 in the arithmetic hierarchy.
- vi) N_ω is definable in N_ω, and hence the language of N_ω contains a truth predicate for N_ω.
- vii) If B is the proof predicate for N_ω then every instance of the scheme $B(<\varphi>) \to \varphi$ is in N_ω.
- viii) If φ is any non-theorem of N_ω $\neg B(<\varphi>)$ is provable in N_ω. In particular, the sentence asserting the non-triviality of N_ω is in N_ω.
- ix) The "Goedel sentence" for N_ω is in N_ω, as its negation.

I have enumerated these properties in such a way as to make a comparison with the properties of each N_n, as enumerated in *IAC*, simple. Such a comparison will show that N_ω shares almost all the properties of its finite cousins, and hence their advantages over *N*. In some ways it is even better. For example, the whole equational part of N_ω is consistent. Moreover, on this account $\exists xx = x+1$ is true. But there is no determinate number, *n*, such that $n = n+1$ is true. This provides a plausible explanation of why we are unable to say what the least inconsistent number is.

The major difference between N_ω and its finite cousins is that each of these is decidable, whilst it itself is π_1, i.e., it is the complement of a recursively enumerable (axiomatisable) set. This affects only one of the considerations proposed in favour of the $N_n s$ against *N* in *IAC*: that concerning the graspability of arithmetic meaning. (Hilbert's Programme must also be considered to fail, though this is not, itself, an objection.)

Each N_n is axiomatic, in fact, decidable, and so it is possible to give a simple account of how it is one gets to know what statements of mathematics mean. We grasp the meaning in grasping the algorithm in question.

Since N_ω is not decidable, or even axiomatic, this simple story is no longer available to us. There is another, however. Even if arithmetic were not decidable, as long as it it were axiomatic, there would be a perfectly intelligible story of how we grasp the nature of mathematical assertions. Understanding a statement would be understanding what constitutes a proof of it. Hence, learning the canons of proof would constitute our grasp of mathematical language. Now N_ω is not axiomatic, but its compliment is, and this would seem just as good. Our grasp of the language is not constituted by our understanding of what it takes to establish the *truth* of assertions, but what it takes to establish their *untruth*. It may well be the case that this account has methodological implications of a Popperean kind for arithmetic. However, we need not pursue these here. The issue is a cognitive one; and this has been answered.

7. *Truth and Proof*

In a nutshell, we can summarise the main point of the previous section thus. It is natural to suppose that the grasp of the meanings of mathematical assertions is constituted by our understanding of their truth conditions. If these could be identified with proof conditions we would have a simple story of how the trick of understanding is turned. Classically, it cannot be identifed, and so there is a problem. (Whether it is decisive, as Dummett (e.g. [1978]) and others have maintained, it is at least a difficult point.) Now, it would seem to make little substantial difference if we identify our grasp of the meaning of arithmetic assertions with a grasp of their *untruth* conditions. But this *can* be identified with provability of a certain kind, at least if N_ω is correct.

The identification of truth (or at least, untruth) with provability, suggests a final, and new, answer to our original question concerning the least inconsistent number. This is a conventionalist one. Conventionalism foreswears an account of the nature of number in favour of a direct account of mathematical truth, or in our case, untruth. This is simply identified with provability, where the standards of proof are taken as constitutive, not as answering to some pre-existing semantic notion. In our case, untruth is simply constituted by provability in a system that delivers the complement of N_ω. This answer to our original problem can be given even by someone who rejects the nominalist account of previous sections. It may, indeed, be the simplest solution to the problem.

8. *Appendix*

In this appendix I outline the proofs of the facts cited in section 6.

- i) holds since each N_n contains N.
- ii) holds since the intersection of a collection of theories is a theory.
- iii) is true since each N_n contains $\exists x x = x+1 \wedge \neg\exists x x = x+1$.
- iv) holds since any (negated) equation that holds in N holds in N_ω, and any (negated) equation that fails in N fails in some N_n, and so in N_ω. That N_ω is unsaturated follows simply form the previous two observations.
- vi) is true since every arithmetic set is definable in N, and so in each N_n, and so in N_ω. That it is a truth predicate follows as in the case for N_n
- vii) and viii) follow straight away.
- ix) follows as in the case for each N_n

This leaves only v). This is proved with the help of a lemma:

Lemma

> Let φ be any formula of the form $\forall x_1 \dots \forall x_m \alpha(x_1 \dots x_m)$, where $\alpha(x_1 \dots x_m)$ is any equation or negated equation. Then $\varphi \in N$ iff $\varphi \in N_\omega$.

Proof

> The 'only if' part follows from i). For the 'if' part, suppose that $\varphi \notin N$. Then for some numerals, $c_1 \dots c_m$, $\alpha(c_1 \dots c_m) \notin N$. Hence, $\alpha(c_1 \dots c_m) \notin N_\omega$, by iv). Hence, $\varphi \notin N_\omega$ since this is a theory.

We can now prove v). To start with, note that N_ω is the intersection of an enumerable set of decidable sets. Hence, it is at worst π_1. Suppose it were decidable. Let Σ be the set of sentences of arithmetic which consist of an equation preceeded by a string of existential quantifiers; we show that Σ is decidable. This is impossible because of the negative solution to Hilbert's 10th problem. (See Bell and Machover [1977], ch 6, sec 16.) The decision procedure is as follows. Suppose that $\sigma \in \Sigma$. σ is false iff its negation is true. But its negation is (logically equivalent) to something of the form $\forall x_1 \dots \forall x_m \alpha(x_1 \dots x_m)$, where $\alpha(x_1 \dots x_m)$ is a negated equation. By the lemma, $\neg\sigma$ is in N iff it is in N_ω. By running the decision procedure for N_ω we can therefore determine whether $\neg\sigma$ is true, and so whether σ is.

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